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The action principle for a system of differential equations

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Abstract
We consider the problem of constructing an action functional for physical systems whose classical equations of motion cannot be directly identified with Euler–Lagrange equations for an action principle. Two ways of constructing the action principle are presented. From simple consideration, we derive the necessary and sufficient conditions for the existence of a multiplier matrix which can endow a prescribed set of second-order differential equations with the structure of the Euler–Lagrange equations. An explicit form of the action is constructed if such a multiplier exists. If a given set of differential equations cannot be derived from an action principle, one can reformulate such a set in an equivalent first-order form which can always be treated as the Euler–Lagrange equations of a certain action. We construct such an action explicitly. There exists an ambiguity (not reduced to a total time derivative) in associating a Lagrange function with a given set of equations. We present a complete description of this ambiguity. The general procedure is illustrated by several examples.

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1. Introduction

The problem of constructing an action functional for a given set of differential equations is known in the literature as the inverse problem of the calculus of variations for Newtonian mechanics. In its classical setting [1] it consists of solving the variational equation

$$\frac{\delta S[q]}{\delta q^i(t)} = g_i, \tag{1}$$

where $g_i(t, q^i, \dot{q}^i, \ldots) = 0$ is some given system of differential equations with respect to unknown functions $q^i(t)$, and $S[q]$ is a local functional to be determined. The condition of locality requires the existence of a function $L(t, q, \dot{q}, \ldots)$ (Lagrangian), such that the functional $S[q]$ (action) would be written as an integral

$$S[q] = \int \mathrm{d}t\, L. \tag{2}$$

In other words, the essence of the inverse problem of the calculus of variations consists of finding a variational principle for a given system of differential equations. This problem has been under consideration for more than a hundred years. As early as 1887, Helmholtz [1] presented a criterion of commutativity for second variational derivatives from which immediately follows the necessary (and with some restrictions also sufficient) condition of solvability of equation (1):

$$\frac{\delta g_i(t)}{\delta q^j(s)} = \frac{\delta g_j(s)}{\delta q^i(t)}. \tag{3}$$

If this condition holds, the system $g_i(t, q^i, \dot{q}^i, \ldots) = 0$ is called the Lagrangian system, if not the system is non-Lagrangian. In 1894, Darboux [2] solved the problem for the one-dimensional case. In 1941, the case of two degrees of freedom was investigated by Douglas [3]; in particular, he presented examples of second-order equations which cannot be obtained from the variational principle. Afterwards, many authors (see, e.g., [6–13] and references therein) investigated this problem for multidimensional systems.

In the present work, we consider the question of the construction of an action principle for a given system of differential equations using the integrating multiplier method [3–9]. The integrating multiplier is a nonsingular matrix which being multiplied by a given set of differential equations reduces this set to a standard Euler–Lagrange form. In section 2, we present a simple derivation for the necessary and sufficient conditions for an integrating multiplier for a system of second-order equations. We also construct the explicit form of the Lagrangian in case an integrating multiplier exists. Then we apply our method for investigating the inverse problem of some simple models. In particular, we construct an action principle for multidimensional dissipative systems. We also consider an example of a linear dynamical system whose equations of motion does not admit an integrating multiplier, and, as a consequence, cannot be obtained from the minimum action principle.

Note that it is always possible to reduce the non-Lagrangian second-order equations of motion to an equivalent set of first-order differential equations. From the Helmholtz criterion (3), we find the necessary and sufficient conditions for the existence of an integrating multiplier for such equations. It turns out that in the first-order formalism an integrating multiplier always exists and can be constructed by means of the solution of the Cauchy problem for the equations in question, and this is presented in section 3 and is partially based on results of the works [10, 13]. Then we construct the action functional explicitly. Thus, we show that systems traditionally called as non-Lagrangian are, in fact, equivalent to some first-order Lagrangian systems. As an example, we construct a first-order action functional for any linear dynamical system.

2. Action functional for a set of second-order equations

2.1. General consideration

Let a system with n degrees of freedom be described by a set of n second-order differential equations of motion, solvable with respect to second-order time derivatives. Suppose such a set has the form

$$\ddot{q}^i - f^i(t, q, \dot{q}) = 0, \qquad i = 1, \ldots, n, \tag{4}$$

where $f^i(t, q, \dot{q})$ are some functions of the indicated arguments, and by dots above we denote time derivatives of the coordinates. Let us construct an action principle for this set. If (4) cannot be directly identified with the Euler–Lagrange equations, one can find an integrating multiplier, i.e., a nonsingular matrix $h_{ij}(t, q, \dot{q})$ that being multiplied by (4)

$$h_{ij}[\ddot{q}^j - f^j(t, q, \dot{q})] = 0 \tag{5}$$

reduces this set to the standard Euler–Lagrange form for some Lagrangian $L(t, q, \dot{q})$,

$$\frac{\partial L}{\partial q^i} - \frac{\partial^2 L}{\partial t \partial \dot{q}^i} - \frac{\partial^2 L}{\partial \dot{q}^i \partial q^j}\dot{q}^j - \frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j}\ddot{q}^j = 0. \tag{6}$$

In order to identify (5) with (6), we need to ensure that

$$\frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j} = h_{ij}, \tag{7}$$

$$\frac{\partial L}{\partial q^i} - \frac{\partial^2 L}{\partial t \partial \dot{q}^i} - \frac{\partial^2 L}{\partial \dot{q}^i \partial q^j}\dot{q}^j = h_{ij} f^j. \tag{8}$$

Provided that an integrating multiplier is known, equations (7) and (8) can be interpreted as a system of equations for a Lagrange function L. We are going to solve the set of equations (7) and (8). Its consistency conditions will give us all the necessary and sufficient conditions for an integrating multiplier. Assuming that L is a smooth function of the indicated arguments, the consistency condition for equation (7) imply that

$$h_{ij} = h_{ji}, \qquad \frac{\partial h_{ij}}{\partial \dot{q}^k} = \frac{\partial h_{kj}}{\partial \dot{q}^i}. \tag{9}$$

If (9) does hold one can solve equation (7). To this end, we remind that the general solution of equation $\partial f/\partial q^i = g_i$, provided the vector g_i is a gradient, is

$$f(q) = \int_0^1 \mathrm{d}s q^i g_i(sq) + c,$$

where c is a constant. Taking the above fact into account, we obtain for L (we do not consider global problems which can arise from nontrivial topology of the configuration space) the following representation:

$$L = K(t, q, \dot{q}) + l_i(t, q)\dot{q}^i + l_0(t, q), \tag{10}$$

where

$$K(t, q, \dot{q}) = \int_0^1 \mathrm{d}a\, \dot{q}^j \left[\int_0^1 \mathrm{d}b\, \dot{q}_1^i h_{ij}(t, q, b\dot{q}_1)\right]_{\dot{q}_1 = a\dot{q}} \tag{11}$$

and $l_0(t, q)$, $l_i(t, q)$ are some functions of the indicated arguments. To find these functions, we use equation (8). Substituting (10) into (8), we get

$$\frac{\partial K}{\partial q^i} - \frac{\partial^2 K}{\partial \dot{q}^i \partial t} - \frac{\partial^2 K}{\partial \dot{q}^i \partial q^j}\dot{q}^j + \left(\frac{\partial l_j}{\partial q^i} - \frac{\partial l_i}{\partial q^j}\right)\dot{q}^j - \frac{\partial l_i}{\partial t} + \frac{\partial l_0}{\partial q^i} = h_{ij} f^j. \tag{12}$$

Differentiating this equation over $\dot{q}^k$, we obtain

$$\frac{\partial l_k}{\partial q^i} - \frac{\partial l_i}{\partial q^k} = L_{ik}, \tag{13}$$

where

$$L_{ik} = \frac{\partial^2 K}{\partial \dot{q}^i \partial q^j} - \frac{\partial^2 K}{\partial \dot{q}^j \partial q^i} + \frac{\partial h_{ik}}{\partial t} + \dot{q}^j \frac{\partial h_{ik}}{\partial q^j} + \frac{\partial}{\partial \dot{q}^k}(h_{ij} f^j). \tag{14}$$

This equation is a differential equation for l_i. The consistency conditions of equation (13) imply that, first of all, the symmetric part of L_{ik} is zero, which can be written as

$$\hat{D}h_{ik} + \frac{1}{2}\left(h_{ij}\frac{\partial f^j}{\partial \dot{q}^k} + h_{kj}\frac{\partial f^j}{\partial \dot{q}^i}\right) = 0, \tag{15}$$

where

$$\hat{D} = \frac{\partial}{\partial t} + \dot{q}^j\frac{\partial}{\partial q^j} + f^j\frac{\partial}{\partial \dot{q}^j}.$$

Using (15), one can rewrite (14) as follows:

$$L_{ik} = \frac{\partial^2 K}{\partial \dot{q}^i \partial q^k} - \frac{\partial^2 K}{\partial \dot{q}^k \partial q^i} + A_{ik}, \qquad A_{ik} = \frac{1}{2}\left(h_{ij}\frac{\partial f^j}{\partial \dot{q}^k} - h_{kj}\frac{\partial f^j}{\partial \dot{q}^i}\right). \tag{16}$$

Next, L_{ik} does not depend on the velocities, i.e., $\partial L_{ik}/\partial \dot{q}^l = 0$, which yields

$$\frac{\partial h_{kl}}{\partial q^i} - \frac{\partial h_{il}}{\partial q^k} = \frac{\partial}{\partial \dot{q}^l}A_{ik}. \tag{17}$$

And, finally, the Jacobi identity

$$\frac{\partial L_{ik}}{\partial q^l} + \frac{\partial L_{kl}}{\partial q^i} + \frac{\partial L_{li}}{\partial q^k} = 0 \Rightarrow \frac{\partial A_{ik}}{\partial q^l} + \frac{\partial A_{kl}}{\partial q^i} + \frac{\partial A_{li}}{\partial q^k} = 0. \tag{18}$$

Provided h_{ij} obeys equations (15), (17) and (18), l_i can be found from equation (13). We remind that the general solution for l_i of equation (13) is given by

$$l_i(t,q) = \int_0^1 \mathrm{d}a\, q^k L_{ki}(t, aq) + \frac{\partial \varphi(t,q)}{\partial q^i}, \tag{19}$$

where $\varphi(t,q)$ is an arbitrary function.

Now from equation (12) we can find l_0; to this end, let us rewrite it as follows:

$$\frac{\partial l_0}{\partial q^i} = m_i, \tag{20}$$

where

$$m_i = h_{ij}f^j - \frac{\partial K}{\partial q^i} + \frac{\partial^2 K}{\partial t \partial \dot{q}^i} + \dot{q}^j\frac{\partial^2 K}{\partial q^j \partial \dot{q}^i} - \dot{q}^j L_{ij} + \frac{\partial l_i}{\partial t}. \tag{21}$$

The consistency conditions of (20) imply that, first, m_i does not depend on the velocities, i.e., $\partial m_i/\partial \dot{q}^k = 0$. This condition is provided by equation (15). And second, the vector m_i must be a gradient:

$$\frac{\partial m_i}{\partial q^k} - \frac{\partial m_k}{\partial q^i} = \frac{\partial A_{ik}}{\partial t} + \dot{q}^j\frac{\partial A_{ik}}{\partial q^j} + \frac{\partial}{\partial q^k}(h_{ij}f^j) - \frac{\partial}{\partial q^i}(h_{kj}f^j) = 0. \tag{22}$$

Taking into account (9), (15) and (17), one gets from (22) the following algebraic condition:

$$h_{ij}B^j_k - h_{kj}B^j_i = 0, \tag{23}$$

where

$$B^i_j = \frac{1}{2}\frac{\partial f^i}{\partial \dot{q}^m}\frac{\partial f^m}{\partial \dot{q}^j} - \hat{D}\frac{\partial f^i}{\partial \dot{q}^j} + 2\frac{\partial f^i}{\partial q^j}.$$

If h_{ij} obeys (23), then from (20) one gets

$$l_0(t,q) = \int_0^1 \mathrm{d}a\, q^k m_k(t, aq) + \frac{\partial \varphi(t,q)}{\partial t} + c(t), \tag{24}$$

where $c(t)$ is an arbitrary function of time.

Thus, we have proved the following statement: iff for a given set of second-order ordinary differential equations (4) there exists a nonsingular matrix $h_{ij}(t, q, \dot{q})$ that obeys equations (9), (15), (17), (18) and (23), then this set can be obtained from the variational principle with the Lagrangian (10), where the functions $K(t, q, \dot{q})$, $l_i(t, q)$ and $l_0(t, q)$ are defined by (11), (19) and (24), respectively, and the functions $\varphi(t, q)$ and $c(t)$ are arbitrary functions of the indicated arguments.

The arbitrariness related to the functions $\varphi(t, q)$ and $c(t)$ enter the Lagrangian (10) via the total time derivative of a function F,

$$F = \varphi(t, q) + \int c(t)\, \mathrm{d}t.$$

Note that an integrating multiplier h_{ij}, and as a consequence the Lagrange function L does exist, but however not for any set of equations (4). In section 3, we consider an example of a dynamical system which does not admit the existence of an integrating multiplier. However, if it exists, it is not unique [13–19], e.g., if the matrix h_{ij} is an integrating multiplier for a certain set (4), it is easy to see that the matrix $\acute{h}_{ij} = ch_{ij}$, where $c \neq 0$ is a constant, is an integrating multiplier as well. Therefore, the Lagrangian (10) leading to the set of equations (4) is not unique because for this set there exist as many inequivalent Lagrangians as integrating multipliers. Lagrangians corresponding to different integrating multipliers are known as *s*-equivalent Lagrangians.

In the one-dimensional case, $\ddot{q} - f(t, q, \dot{q}) = 0$, an integrating multiplier is a non-vanishing function $h(t, q, \dot{q})$ that obeys the equation

$$\frac{\partial h}{\partial t} + \dot{q}\frac{\partial h}{\partial q} + \frac{\partial}{\partial \dot{q}}(fh) = 0. \tag{25}$$

This is a first-order partial differential equation which obviously has a solution for any f and initial condition $h(t = 0, q, \dot{q}) = h_0(q, \dot{q})$. As we can see, an answer to the question whether there exists a solution of the inverse problem of the calculus of variations depends on the number of degrees of freedom n. For $n = 1$, the answer is always positive, and there exist as many inequivalent Lagrangians as functions $h_0(q, \dot{q})$ of two variables. For $n \geqslant 2$, the answer is generally negative.

2.2. Examples

In this section, we consider the possibility of constructing an action principle for some examples of dynamical systems. First of all, let us consider dissipative systems. Suppose we have an ideal system with the Lagrangian

$$L_0 = \frac{\dot{q}^2}{2} + V(q), \qquad q = \{q^i\}, \qquad i = 1, \ldots, n. \tag{26}$$

Let us consider the case when besides the potential conservative force $F^i = \frac{\partial V}{\partial q^i}$ there exist a friction force

$$F^i_{\text{fric}} = \alpha \dot{q}^i, \tag{27}$$

where α is a phenomenological friction coefficient which in general can depend on time. The equations of motion for such a system have the form

$$\ddot{q}^i = \frac{\partial V}{\partial q^i} + \alpha \dot{q}^i. \tag{28}$$

These equations are non-Lagrangian, but for this set it is possible to find an integrating multiplier. In the simplest case, when it does not depend on coordinates and velocities, it has the form

$$h_{ij} = \mathrm{e}^{-2\int \alpha \mathrm{d}t} h^0_{ij}, \tag{29}$$

where h^0_{ij} is an arbitrary, symmetric, nonsingular, constant matrix commuting with the matrix $V_{ij} = \partial^2 V/\partial q^i \partial q^j$. Using the statement of the previous section, we obtain the following Lagrangian:

$$L = \frac{1}{2}\dot{q}^i h_{ij}\dot{q}^j + \int_0^1 q^i h_{ij} \frac{\partial V(s\vec{q})}{\partial q^j} \,\mathrm{d}s. \tag{30}$$

If one sets $h^0_{ij} = \delta_{ij}$, the Lagrangian (30) can be rewritten as

$$L = \mathrm{e}^{-2\int \alpha \mathrm{d}t} L_0. \tag{31}$$

Note that once the friction coefficient goes to zero, the Lagrangian (31) transforms into the initial Lagrangian (26).

Let us now consider the case when the potential in the initial Lagrangian is linear in velocities. For simplicity, we consider the two-dimensional case:

$$L_0 = \tfrac{1}{2}(\dot{x}^2 + \dot{y}^2 + \beta(\dot{x}y - \dot{y}x)). \tag{32}$$

Let us consider this system in the presence of the dissipative force (30). The equations of motion will have the form

$$\ddot{x} = \alpha\dot{x} - \beta\dot{y}, \qquad \ddot{y} = \beta\dot{x} + \alpha\dot{y}. \tag{33}$$

As was shown in [12], this system describes a moving charged particle in a uniform magnetic field with radiation friction. In this case,

$$B^i_j = \begin{pmatrix} \alpha^2 - \beta^2 & -2\alpha\beta \\ 2\alpha\beta & \alpha^2 - \beta^2 \end{pmatrix}$$

and from equation (23) one immediately gets

$$\mathrm{tr}(h_{ij}) = h_{11} + h_{22} = 0. \tag{34}$$

It is then easy to find that the general solution of equations (9), (15), (17) and (18) is defined by an arbitrary function $\phi(\zeta, \eta)$ and has the form

$$h_{ij} = \begin{pmatrix} F + \bar{F} & \mathrm{i}\left(F - \bar{F}\right) \\ \mathrm{i}\left(F - \bar{F}\right) & -\left(F + \bar{F}\right) \end{pmatrix}, \tag{35}$$

where $F = \phi(\dot{\xi}\mathrm{e}^{-\gamma t}, \dot{\xi} - \alpha\xi)\,\mathrm{e}^{-\gamma t}$, $\xi = x + \mathrm{i}y$, $\gamma = \alpha + \mathrm{i}\beta$ and the bar denotes a complex conjugation.

The simplest real solution can be found if we put $\phi = 1/\zeta$. We have

$$h_{ij} = \frac{2}{\dot{x}^2 + \dot{y}^2} \begin{pmatrix} \dot{x} & \dot{y} \\ \dot{y} & -\dot{x} \end{pmatrix}. \tag{36}$$

Using formulae (10), (11), (19) and (24), we find the following Lagrangian:

$$L = \frac{1}{2}\dot{x}\ln(\dot{x}^2 + \dot{y}^2) + \dot{y}\arctan\left(\frac{\dot{x}}{\dot{y}}\right) + \alpha x - \beta y. \tag{37}$$

The corresponding Euler–Lagrange equations

$$\frac{\ddot{x}\dot{x} + \ddot{y}\dot{y}}{\dot{x}^2 + \dot{y}^2} = \alpha, \qquad \frac{\ddot{x}\dot{x} - \ddot{y}\dot{y}}{\dot{x}^2 + \dot{y}^2} = \beta \tag{38}$$

are equivalent to the initial ones (33), with the exception of the point $\dot{x} = \dot{y} = 0$. Thus, we can see that in this case the inverse problem of the calculus of variations is solvable. Unfortunately, neither the Lagrangian (37) nor any other Lagrangian constructed by the matrix (35) in the limit $\alpha \to 0$ transforms into the initial Lagrangian (32), modulo a total time derivative. This is because, according to the algebraic condition (34), the trace of the Hessian matrix of any

Lagrangian for the set of equations (33) must be equal to zero and this property holds true after the limit $\alpha \to 0$ is taken. On the other hand, the trace of the Hessian matrix of the Lagrangian L_0 in (32) is equal to 2. This contradiction proves the statement.

Finally, we consider the example of a dynamical system for which an integrating multiplier and, consequently, the possibility of the Lagrangian description does not exist. Douglas [3] showed that the set of second-order equations

$$\ddot{x} + \dot{y} = 0, \qquad \ddot{y} + y = 0$$

does not admit an integrating multiplier. Let us prove this. To this end, let us assume the opposite, namely, let there be a non-degenerate matrix h_{ij} that obeys equations (9), (15), (17), (18) and (23). Then from the algebraic equation (23) it follows that h_{ij} must be diagonal ($h_{12} = h_{21} = 0$), since in this case

$$B^i_j = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}.$$

Then, from condition (15) we immediately obtain $h_{11} = 0$, and arrive at a contradiction, $\det h_{ij} = 0$.

Thus, we can see that an action functional in the second-order formalism cannot be constructed for some sets of differential equations. Nevertheless, as we show in the following section, it is always possible to construct an action principle for the equivalent set of first-order equations.

3. An action principle in the first-order form

Let a system with n degrees of freedom be described by a set of n non-Lagrangian second-order differential equations of motion. To construct an action principle, we replace these equations (which is always possible to do by introducing n additional variables, e.g., $p_i = \dot{q}^i$) by an equivalent set of $2n$ first-order differential equations, solvable with respect to time derivatives. Suppose such a set has the form

$$\dot{x}^\alpha = f^\alpha(t, x), \qquad x^\alpha = (q^i, p_i), \qquad \alpha = 1, \ldots, 2n, \tag{39}$$

where $f^\alpha(t, x)$ are some functions of the indicated arguments and by dots above we denote time derivatives of coordinates. Let us construct an action principle for this set. If (39) cannot be directly identified with the Euler–Lagrange equations, then one can find an integrating multiplier, i.e., a nonsingular matrix Ω[3] which reduces the initial set of differential equations (39) to a variational derivative:

$$g_\alpha[t] = \Omega_{\alpha\beta}(\dot{x}^\beta - f^\beta(t, x)) = \frac{\delta S}{\delta x^\alpha} = 0. \tag{40}$$

Since $g_\alpha[t]$ is a variational derivative it must obey the Helmholtz criterion [1]

$$\frac{\delta g_\alpha[t]}{\delta x^\beta(s)} = \frac{\delta g_\beta[s]}{\delta x^\alpha(t)}. \tag{41}$$

We will use this condition to find an integrating multiplier Ω. In the general case one can assume that Ω depends on time t, coordinates x^α and time derivatives up to order m (m is a natural number), i.e., $g_\alpha[t] = g_\alpha(t, x, \ldots, x^{(m)})$. Having in mind this form of g_α, one rewrites (41) as

$$\sum_{i=0}^{m} \frac{\partial g_\alpha[t]}{\partial x^{\beta(i)}} \delta^{(i)}(t - s) = \sum_{j=0}^{m} \frac{\partial g_\beta[s]}{\partial x^{\alpha(j)}} \delta^{(j)}(s - t), \tag{42}$$

[3] We denote by Ω the integrating multiplier for the first-order equations.

since

$$\frac{\delta}{\delta x^{\beta}(s)} x^{\alpha(k)}(t) = \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{k} \frac{\delta x^{\alpha}(t)}{\delta x^{\beta}(s)} = \delta^{\alpha}_{\beta} \delta^{(k)}(t-s), \qquad k = 0, 1, \ldots.$$

Differentiating the identity $f(t)\delta(t-s) = f(s)\,\delta(s-t)$ over t, one finds

$$f(s)\delta^{(k)}(s-t) = (-1)^{k} \sum_{l=0}^{k} C^{k}_{l} f^{(l)}(t) \delta^{(k-l)}(t-s), \qquad C^{k}_{l} = \frac{k!}{(k-l)!}. \tag{43}$$

Using (43), we rewrite (41) as

$$\sum_{i=0}^{m} \frac{\partial g_{\alpha}[t]}{\partial x^{\beta(i)}} \delta^{(i)}(t-s) = \sum_{j=0}^{m} (-1)^{j} \sum_{l=0}^{j} C^{j}_{l} \left[\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{l} \frac{\partial g_{\beta}[t]}{\partial x^{\alpha(j)}}\right] \delta^{(j-l)}(t-s). \tag{44}$$

Comparing in (44) the coefficient for $\delta^{(k)}(t-s)$, one gets equations for $g_{\alpha}(t, x, \ldots, x^{(m)})$. When $k = 0$, we have

$$\frac{\partial g_{\alpha}}{\partial x^{\beta}} - \sum_{j=0}^{m} (-1)^{j} \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{l} \frac{\partial g_{\beta}}{\partial x^{\alpha(j)}} = 0. \tag{45}$$

Since $g_{\alpha}[t]$ depends only on derivatives up to order m, the coefficient of the higher derivative $x^{\alpha(2m)}$ in equation (45) must vanish. This coefficient is $(-1)^{m} \partial^{2} g_{\beta} / \partial x^{\alpha(m)} \partial x^{\gamma(m)}$, which means that $g_{\alpha}[t]$ must be linear on the derivatives of order m, i.e.,

$$g_{\alpha}[t] = a_{\alpha\beta}(t, x, \ldots, x^{(m-1)}) x^{\beta(m)} + b_{\alpha}(t, x, \ldots, x^{(m-1)}),$$

where $a_{\alpha\beta}$ and b_{α} are some functions. Since Ω is a nonsingular matrix, (40) should be a system of first-order equations, i.e., we have $m = 1$ and $\Omega = \Omega(t, x)$.

Now comparing the coefficient of $\delta^{(1)}(t-s)$ in (44), we obtain

$$\Omega_{\alpha\beta} = -\Omega_{\beta\alpha}. \tag{46}$$

Then from (45), we have

$$\partial_{\beta}(\Omega_{\alpha\gamma} f^{\gamma}) - \partial_{\alpha}(\Omega_{\beta\gamma} f^{\gamma}) + \partial_{t} \Omega_{\alpha\beta} + \dot{x}^{\gamma} (\partial_{\beta} \Omega_{\alpha\gamma} - \partial_{\alpha} \Omega_{\beta\gamma} + \partial_{\gamma} \Omega_{\beta\alpha}) = 0.$$

Since Ω does not depend on $\dot{x}$, one gets the following equations for Ω:

$$\partial_{\alpha} \Omega_{\beta\gamma} + \partial_{\beta} \Omega_{\gamma\alpha} + \partial_{\gamma} \Omega_{\alpha\beta} = 0 \tag{47}$$

and

$$\partial_{t} \Omega_{\alpha\beta} + \pounds_{f} \Omega_{\alpha\beta} = 0, \tag{48}$$

where $\pounds_{f} \Omega_{\alpha\beta}$ is the Lie derivative of $\Omega_{\alpha\beta}$ along the vector field f^{γ} and $\partial_{\alpha} = \partial/\partial x^{\alpha}$, $\partial_{t} = \partial/\partial t$. Thus, we see that for a set of first-order equations an integrating multiplier is a nonsingular matrix which depends only on time t and coordinates x^{α}, and obeys the conditions (46)–(48).

Let us analyse equations (46)–(48) for the matrix $\Omega_{\alpha\beta}$ following our work [13]. It is known that the general solution $\Omega_{\alpha\beta}$ of equation (48) can be constructed with the help of a solution of the Cauchy problem for equations (39). Suppose that such a solution is known:

$$x^{\alpha} = \varphi^{\alpha}(t, x_{(0)}), \qquad x^{\alpha}_{(0)} = \varphi^{\alpha}(0, x_{(0)}) \tag{49}$$

is a solution of equations (39) for any $x_{(0)} = \left(x^{\alpha}_{(0)}\right)$ and $\chi^{\alpha}(t, x)$ is the inverse function with respect to $\varphi^{\alpha}(t, x_{(0)})$, i.e.,

$$x^{\alpha} = \varphi^{\alpha}(t, x_{(0)}) \Longrightarrow x^{\alpha}_{(0)} = \chi^{\alpha}(t, x), \qquad x^{\alpha} \equiv \varphi^{\alpha}(t, \chi^{\alpha}), \qquad \partial_{\alpha} \chi^{\gamma}|_{t=0} = \delta^{\alpha}_{\gamma}. \tag{50}$$

Then

$$\Omega_{\alpha\beta}(t, x) = \partial_{\alpha} \chi^{\gamma} \Omega^{(0)}_{\gamma\delta}(\chi) \partial_{\beta} \chi^{\delta}, \tag{51}$$

where the matrix $\Omega^{(0)}_{\alpha\beta}$ is the initial condition for $\Omega_{\alpha\beta}$,

$$\Omega_{\alpha\beta}(t,x)|_{t=0} = \Omega^{(0)}_{\alpha\beta}(x).$$

One can see that by choosing the matrix $\Omega^{(0)}_{\alpha\beta}(x)$ to be nonsingular and subject to the Jacobi identity, we guarantee the fulfilment of the same conditions for the complete matrix $\Omega_{\alpha\beta}(t,x)$, since components of the latter are given by a change of variables (51).

Thus, we see that for any set of first-order equations (39), an integrating multiplier always exists, i.e., there always exists a Lagrangian $L(t,x,\dot{x})$ which has the set of equations

$$\Omega_{\alpha\beta}(\dot{x}^\beta - f^\beta(t,x)) = 0 \tag{52}$$

as its Euler–Lagrange equations. It is easy to see that this Lagrangian is linear in the first-order derivative $\dot{x}^\alpha$ since equations (52) do not contain second-order derivatives, i.e., the corresponding term $\partial^2 L/\partial\dot{x}^\alpha\partial\dot{x}^\beta$ vanishes. The general form of this Lagrangian is

$$L = J_\alpha \dot{x}^\alpha - H, \tag{53}$$

where $J_\alpha = J_\alpha(t,x)$ and $H = H(t,x)$ are some functions of the indicated arguments. The Euler–Lagrange equations corresponding to (53) are

$$\frac{\delta S}{\delta x} = \frac{\partial L}{\partial x} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{x}} = 0 \Longrightarrow -\partial_\alpha H - \partial_t J_\alpha + (\partial_\alpha J_\beta - \partial_\beta J_\alpha)\dot{x}^\beta = 0. \tag{54}$$

Comparing equations (52) and (54), one gets[4]

$$\Omega_{\alpha\beta} = \partial_\alpha J_\beta - \partial_\beta J_\alpha, \tag{55}$$

and

$$\Omega_{\alpha\beta} f^\beta - \partial_t J_\alpha = \partial_\alpha H. \tag{56}$$

The functions J_α and H can be found from conditions (55) and (56) if the matrix $\Omega_{\alpha\beta}$ is given. One can see that consistency conditions for these equations exactly give us equations (46) and (48) for an integrating multiplier $\Omega_{\alpha\beta}$. We recall that the general solution $J_\alpha(t,x)$ of equation (55), provided that $\Omega_{\alpha\beta}$ is a given antisymmetric matrix that obeys the Jacobi identity, is given by

$$J_\alpha(t,x) = \int_0^1 x^\beta \Omega_{\beta\alpha}(t,sx)s\,\mathrm{d}s + \partial_\alpha\varphi(t,x), \tag{57}$$

where $\varphi(x)$ is an arbitrary function. Substituting (51) into (57), we obtain

$$J_\alpha(t,y) = \int_0^1 y^\beta\left[\partial_\alpha\chi^\gamma\Omega^{(0)}_{\gamma\delta}(\chi)\partial_\beta\chi^\delta\right]\Big|_{x=sy} s\mathrm{d}s + \partial_\alpha\varphi(t,y). \tag{58}$$

Equation (58) describes all the ambiguity (an arbitrary symplectic matrix $\Omega^{(0)}_{\gamma\delta}$ and an arbitrary function $\varphi(t,x)$) in constructing the term $J_\alpha(t,x)$ of the Lagrange function (53).

To restore the term H in the Lagrange function (53), we need to solve equation (56) with respect to H. To this end, we recall that the general solution of the equation $\partial_i f = g_i$, provided a vector g_i is a gradient, is given by

$$f(x) = \int_0^1 \mathrm{d}s x^i g_i(sx) + c,$$

[4] As a more abstract argument in favour of such a form of Ω we invoke the Poincare lemma, according to which any closed form is locally exact, but here we do not treat the global problems which can arise from the nontrivial topology of x^α-space.

where c is a constant. Taking the above into account, we obtain for H the following representation:

$$H(t,x) = \int_0^1 \mathrm{d}s\, x^\beta [\Omega_{\beta\alpha}(t,sx) f^\alpha(t,sx) - \partial_t J_\beta(t,sx)] + c(t), \tag{59}$$

where $c(t)$ is an arbitrary function of time and $\Omega_{\beta\alpha}$ and J_β are given by (51) and (58), respectively. All the arbitrariness in constructing H is thus due to the arbitrary symplectic matrix $\Omega^{(0)}_{\gamma\delta}$ and the arbitrary functions $\varphi(t,x)$ entering the expressions for $\Omega_{\beta\alpha}$ and J_β and the arbitrary functions $c(t)$.

We can see that there exists a family of Lagrangians (53) which lead to the same equations of motion (39). It is easy to see that actions with the same $\Omega^{(0)}_{\gamma\delta}$ but different functions $\varphi(t,x)$ and $c(t)$ differ by a total time derivative (we call such a difference trivial). A difference in Lagrange functions related to different choice of symplectic matrices $\Omega^{(0)}_{\alpha\beta}$ is not trivial. The corresponding Lagrangians are known as s-equivalent Lagrangians.

As an example, let us consider a theory with equations of motion of the form[5]

$$\dot{x} = A(t)x + j(t). \tag{60}$$

An action principle for such a theory can be constructed (see [13]) following the above described manner.

The solution of the Cauchy problem for equations (60) reads

$$x(t) = \Gamma(t)x_{(0)} + \gamma(t), \tag{61}$$

where the matrix $\Gamma(t)$ is the fundamental solution of (60), i.e.,

$$\dot{\Gamma} = A\Gamma, \qquad \Gamma(0) = 1 \tag{62}$$

and $\gamma(t)$ is a partial solution of (60). Then following (51), we construct the matrix Ω [6],

$$\Omega = \Lambda^T \Omega^{(0)} \Lambda, \qquad \Lambda = \Gamma^{-1} \tag{63}$$

and find the functions J and H according to (58) and (59),

$$J = \tfrac{1}{2} x\Omega, \qquad H = \tfrac{1}{2} xBx - Cx, \tag{64}$$

where

$$B = \tfrac{1}{2}(\Omega A - A^{\mathrm{T}}\Omega), \qquad C = \Omega j. \tag{65}$$

Thus, the action functional for the general quadratic theory is

$$S[x] = \frac{1}{2} \int \mathrm{d}t\, (x\Omega\dot{x} - xBx - 2Cx)\,. \tag{66}$$

In conclusion, we note that it is always possible to construct a Lagrangian action for any set of non-Lagrangian equations in an extended configuration space following a simple idea first proposed by Bateman [20]. Such a Lagrangian has the form of a sum of the initial equations of motion being multiplied by the corresponding Lagrangian multipliers and new variables. The Euler–Lagrange equations for such an action contain besides the initial equations some new equations of motion for the Lagrange multipliers. In such an approach, one has to think how to interpret the new variables already on the classical level. Additional difficulties (indefinite metric) can appear in course of the quantization [21–24].

[5] Here we use matrix notation, $x = (x^\alpha)$, $A(t) = (A(t)^\alpha_\beta)$, $j(t) = (j(t)^\alpha)$, $\alpha, \beta = 1, \ldots, 2n$.

[6] For simplicity we choose the matrix $\Omega^{(0)}$ to be a constant nonsingular antisymmetric matrix.

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